Joint FRISP/WAIS (Forum for Research in Ice Shelf Processes / West Antarctic Ice Sheet) workshops, Pack Forest Conference Center, nr. Mt. Rainier, Washington, Sun 27-29 Sept. '09

# *Progress in Modeling Sheet-Flow Outburst Flooding (as hydraulic fracture by turbulently flowing water)*

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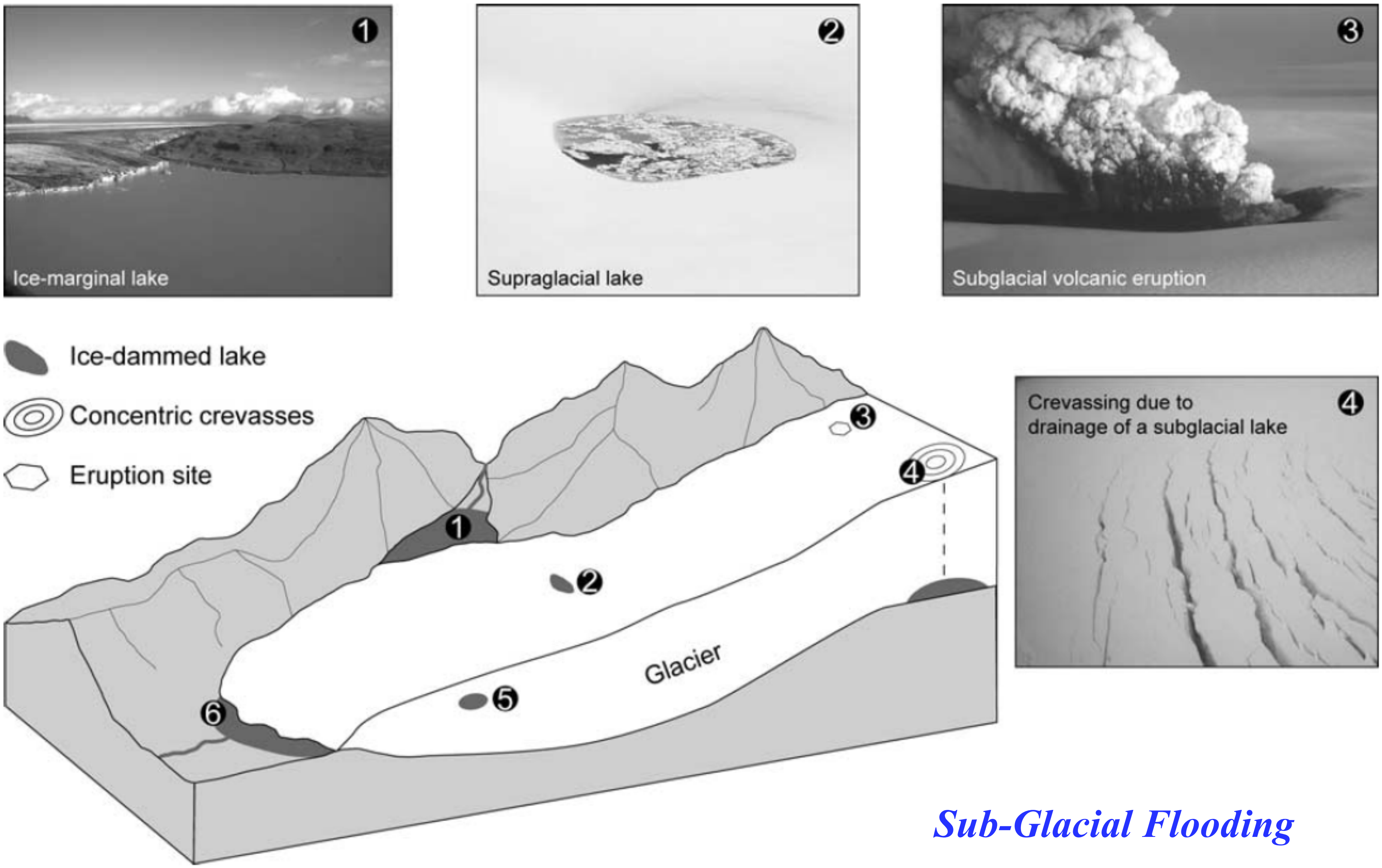
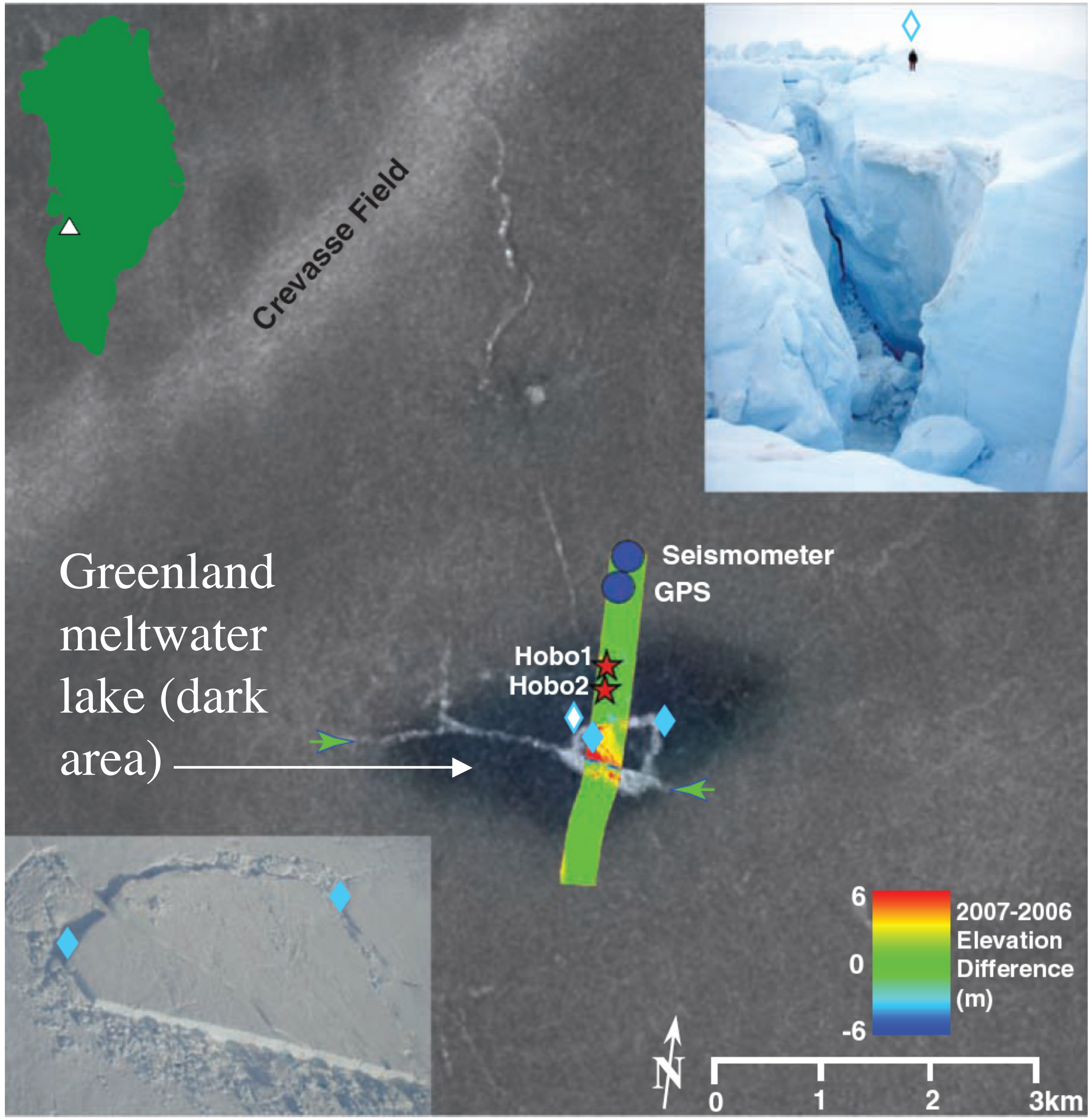



*Sub-Glacial Flooding*

**Figure 1.** Reservoir sites and meltwater sources for jökulhlaups.

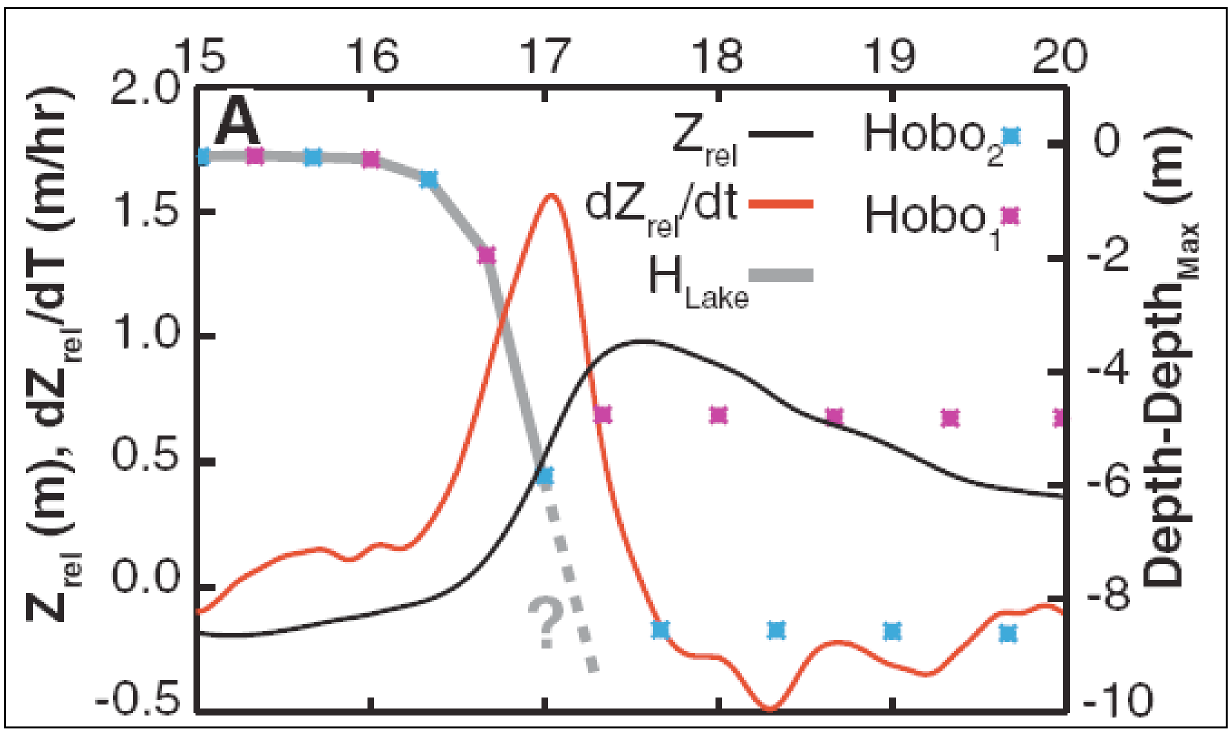
from: Roberts, M. J. (2005), Jökulhlaups: A reassessment of
floodwater flow through glaciers, *Rev. Geophys.,* 43, RG1002.

Das, Joughin, Behn, Howat, King, Lizarralde & Bhatia, ***Fracture Propagation to the Base of the Greenland Ice Sheet During Supraglacial Lake Drainage***, *Science*, May 2008

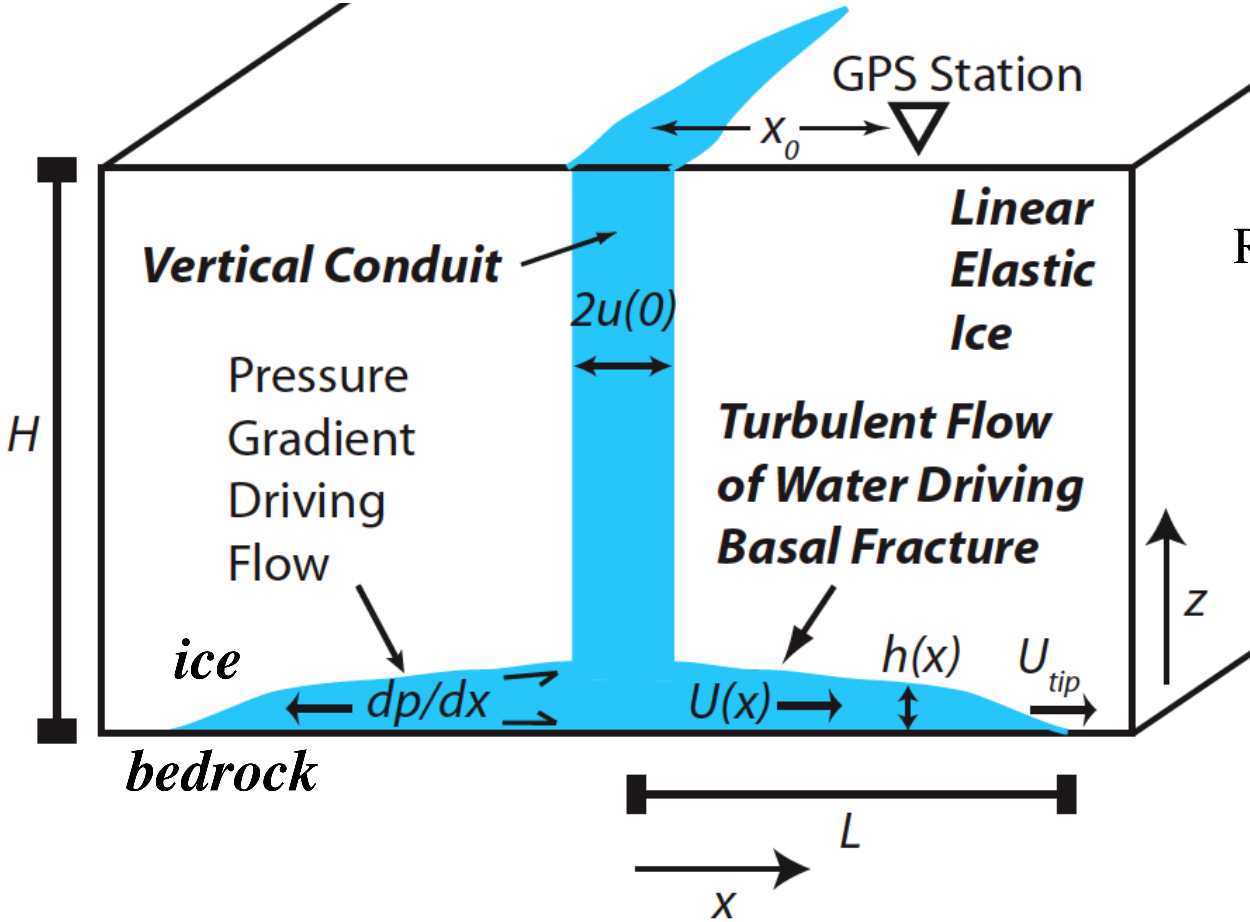


- Supra-glacial meltwater lake began filling July 2006
- Maximum ~0:00 29 July 2006, volume 0.044 km$^3$, area 5.6 km$^2$
- Level slowly/steadily falls, 15 mm/hr
- Rapid from 16:00-17:30, max $Q$ > $10^4$ m$^3$/s, avg. $Q \sim 8.7\times10^3$ m$^3$/s

  [Compare, Niagra Falls $Q \sim 6\times10^3$ m$^3$/s]




(Das et al., *Sci.*, 2008)

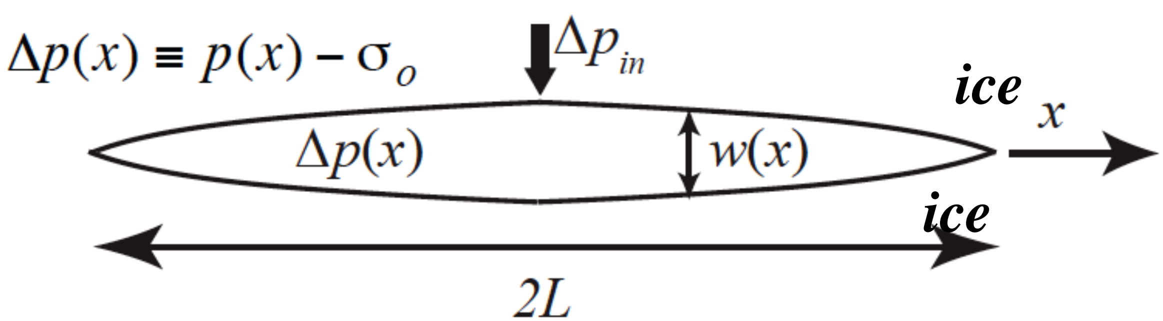



$$\mathrm{Re} = \frac{Uh}{\nu} = \frac{Uh}{2\times10^{-6}\,\mathrm{m}^2/\mathrm{s}}$$

$$= \frac{Q}{2\times10^{-6}\,\mathrm{m}^2/\mathrm{s}\ \mathrm{Width}}$$

$$\approx \frac{10^3\text{-}10^4\,\mathrm{m}^3/\mathrm{s}}{2\times10^{-6}\,\mathrm{m}^2/\mathrm{s}\ 5\mathrm{km}}$$

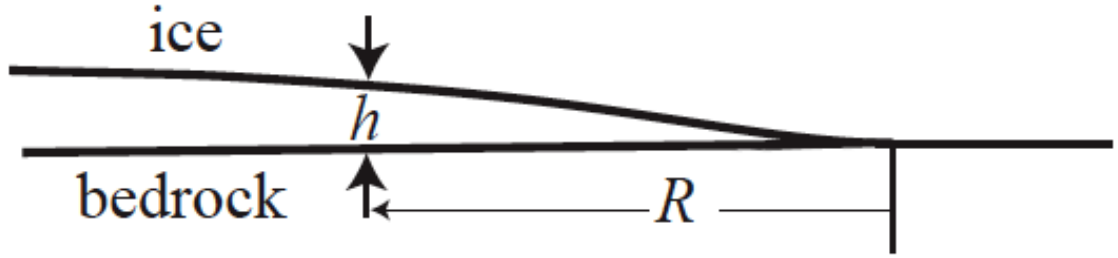
$$= 10^5\text{-}10^6$$



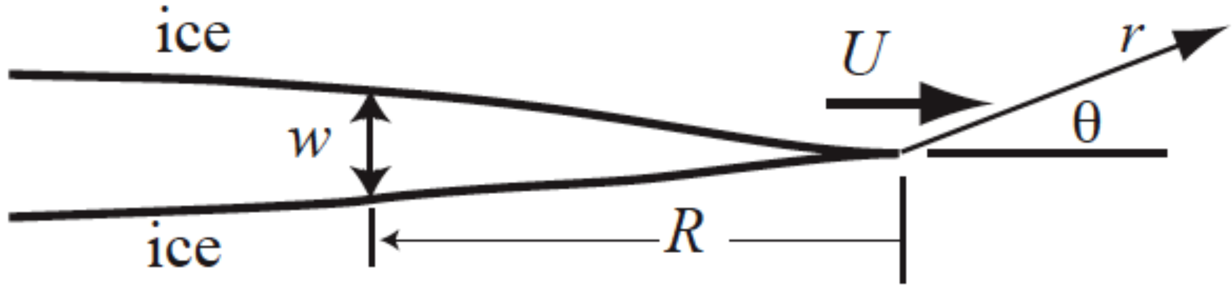

**Results most complete for small $L/H$ range; not the full range of interest!**

## Elasticity modeling:

***Actual inhomogeneous (ice/bedrock) system:***




***Crack in homogeneous (ice) solid:***




For given pressure loading along crack surface, we calculate opening $w$, and then write $h = \xi w$ with

$$\xi = \frac{1 + E'_{ice} / E'_{bedrock}}{2} \approx 0.55, \text{ where } E' = \frac{E}{1-\nu^2}$$

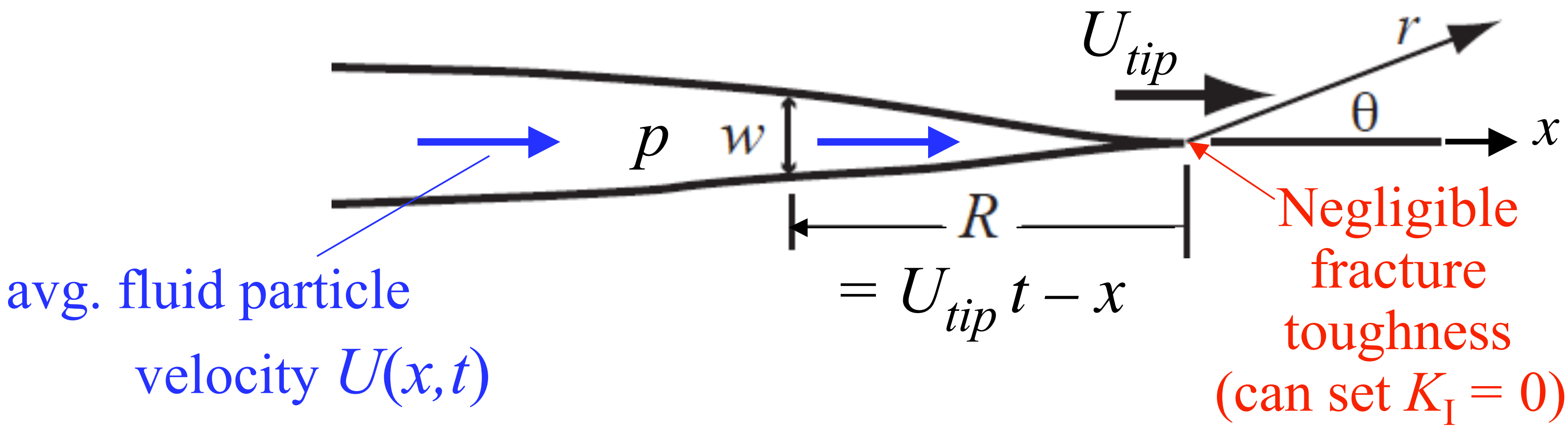
$p(x,t)$ = pressure within liquid in crack space

$\sigma_o$ = overburden pressure exerted by ice sheet

$$p(x,t) - \sigma_o = \frac{E'}{4\pi} \int_{-L(t)}^{+L(t)} \frac{\partial w(s,t)}{\partial s} \frac{ds}{x-s} \quad ; \quad E' = E'_{ice}$$

$$p(x,t) - \sigma_o = \frac{E'}{4\pi\xi} \int_{-L(t)}^{+L(t)} \frac{\partial h(s,t)}{\partial s} \frac{ds}{x-s} \text{ , with } h = \xi w$$

Modifying analysis by Desroches, Detournay, Lenoach, Papanastasiou, Pearson, Thiercelin and Cheng (***The crack tip region in hydraulic fracturing***, *Proc. R. Soc. Lond.*, 1994) for turbulent range:




*Results valid* ***asymptotically*** ($R \rightarrow 0$), *as well as for all R assuming* ***steady state*** *solution in infinite solid:*

$$w(x,t) = w(R) = w(U_{tip}t - x)$$

- Conservation of fluid volume:

$$\frac{\partial(wU)}{\partial x} + \frac{\partial w}{\partial t} = 0 \quad \text{with} \quad w(x,t) = w(R) \;\Rightarrow\; U(x,t) = U_{tip}$$

Schulson and Duval , *Creep and Fracture of Ice*, 2009

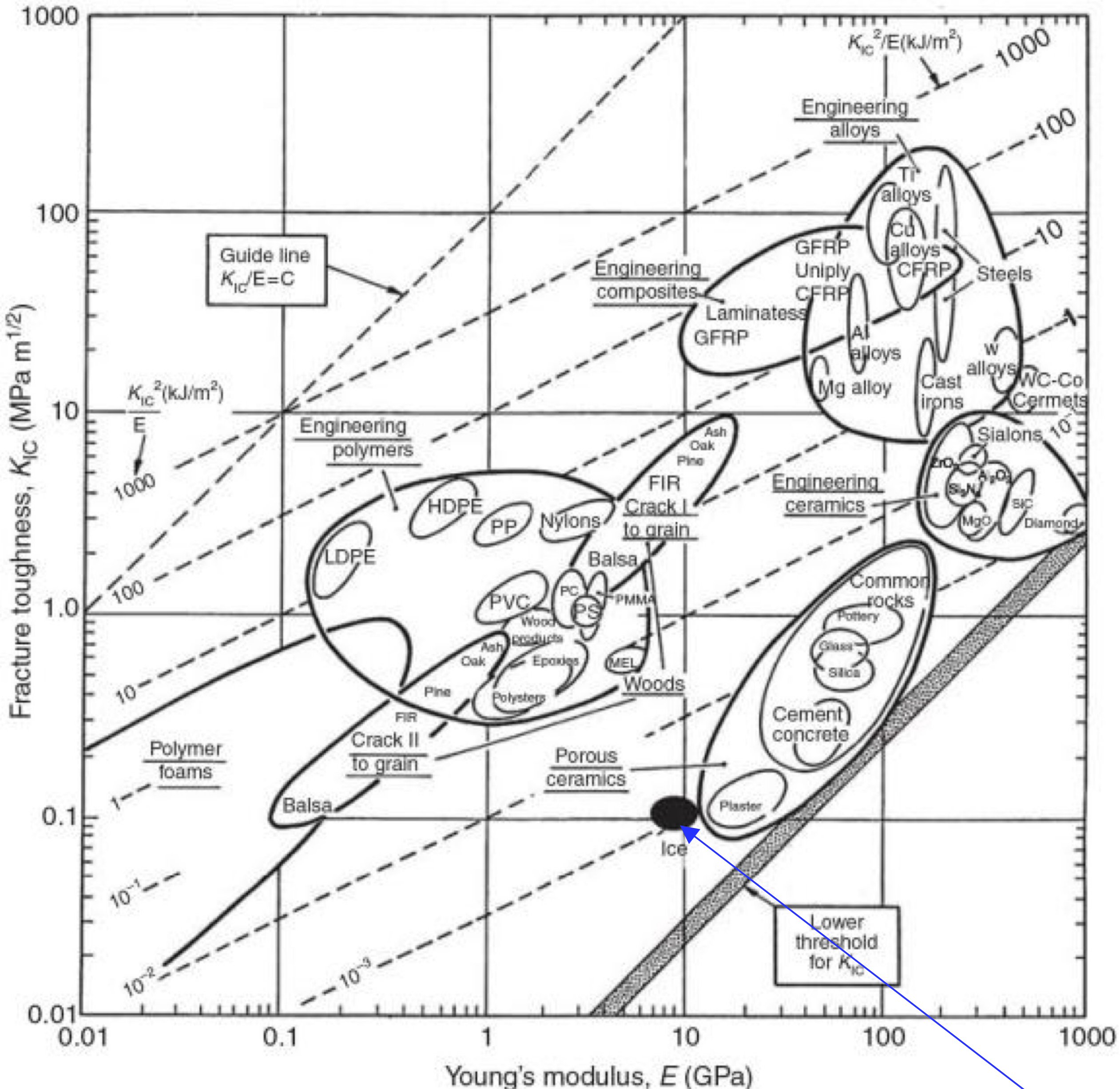
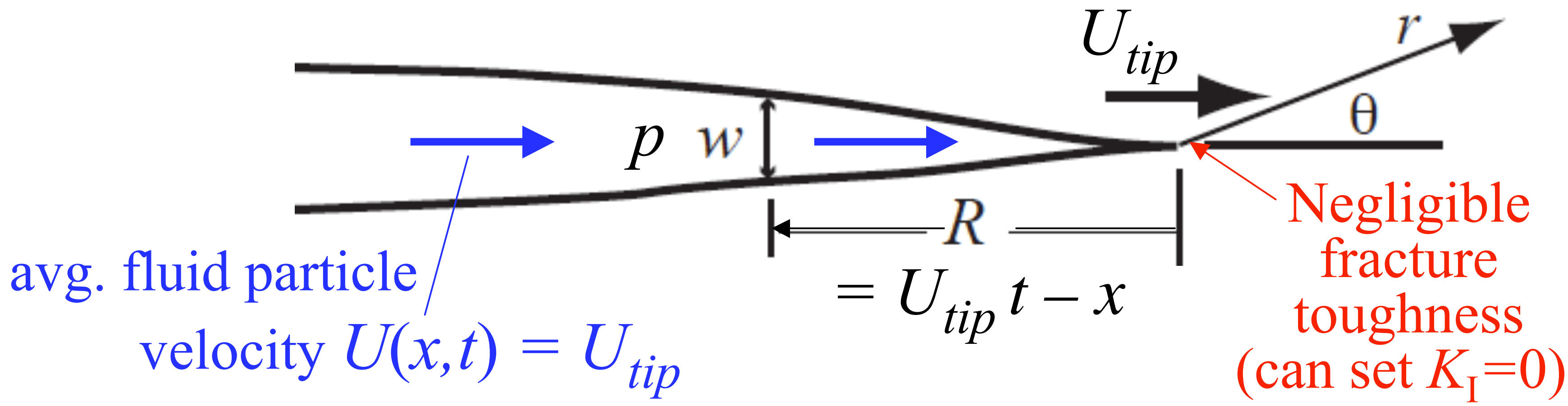



Figure 9.6. Fracture toughness vs. density for a variety of materials, including ice. (From Ashby, 1989.)

For representative $p_{inlet}$ and $L$:

$$\frac{K_{\mathrm{Ic}}}{(p_{inlet}-\sigma_o)\sqrt{\pi L}} << 1$$




- Elasticity theory: Relates $w(R)$ and $p(R)$ distributions.
- Turbulent fluid flow with Manning-Strickler resistance:

Wall roughness

$$\tau_{wall} = \frac{1}{2}\xi w \frac{dp}{dR}\,, \quad \tau_{wall} = \frac{f}{4} \times \frac{1}{2}\rho U^2\,, \quad f = 0.143\left(\frac{k}{\xi w}\right)^{1/3}$$

- Form of solution near fracture tip:

$$p = C_1 - A' R^{-1/7},\ w = 2.247\frac{A'}{E'}R^{6/7},\ A' = 0.489 k^{1/7} E'^{4/7}(\rho U_{tip}^2)^{3/7}$$

$$[\sigma_{yy} + C_1,\ \sigma_{yx},\ \sigma_{xx} + C_2] = A' r^{-1/7}[F_{yy}(\theta),\ F_{yx}(\theta),\ F_{xx}(\theta)]$$

Gioia & Chakraborty [*PRL*, 2006] replot of Johann Nikuradse [1933] pipe-flow data

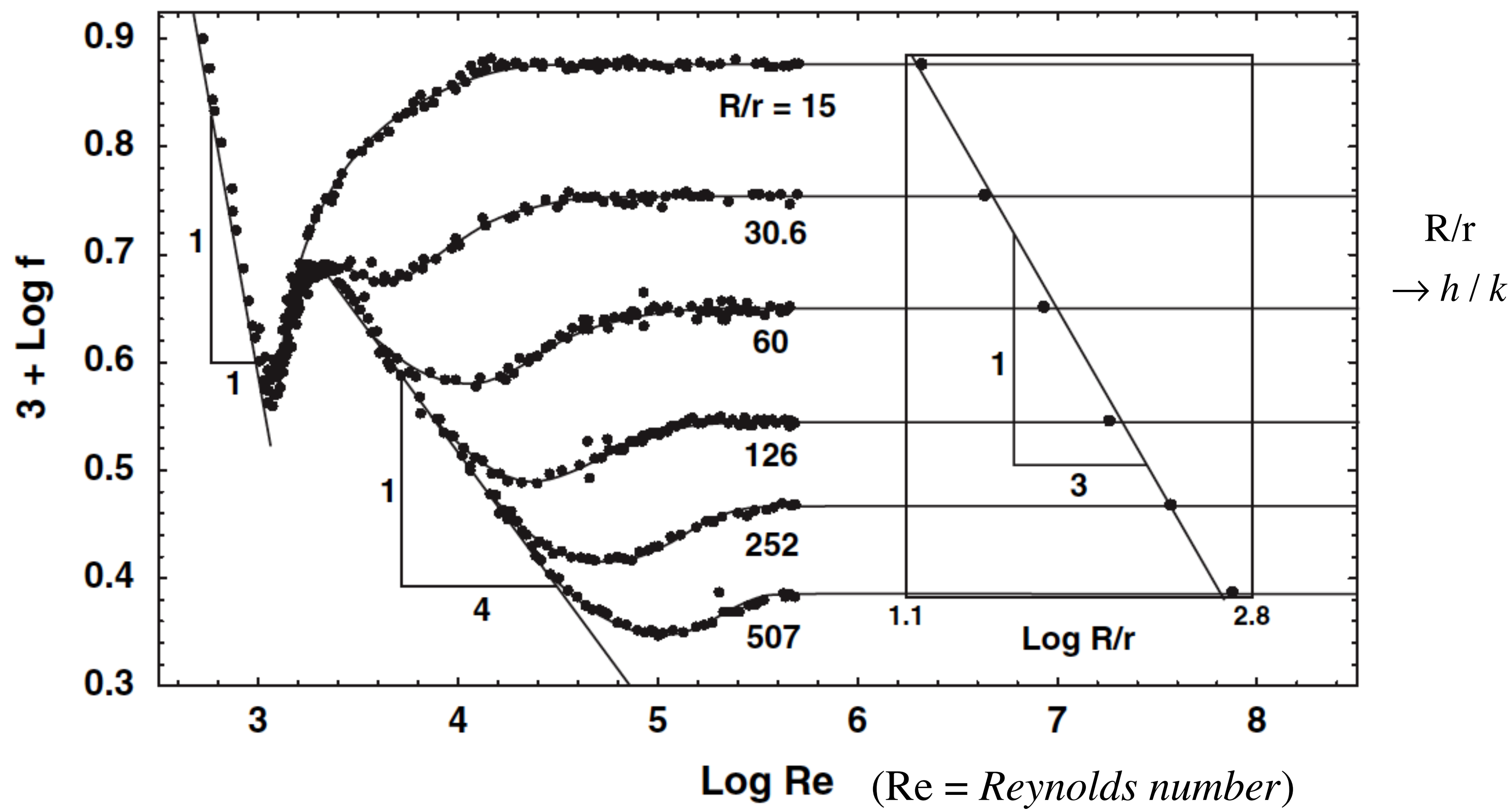
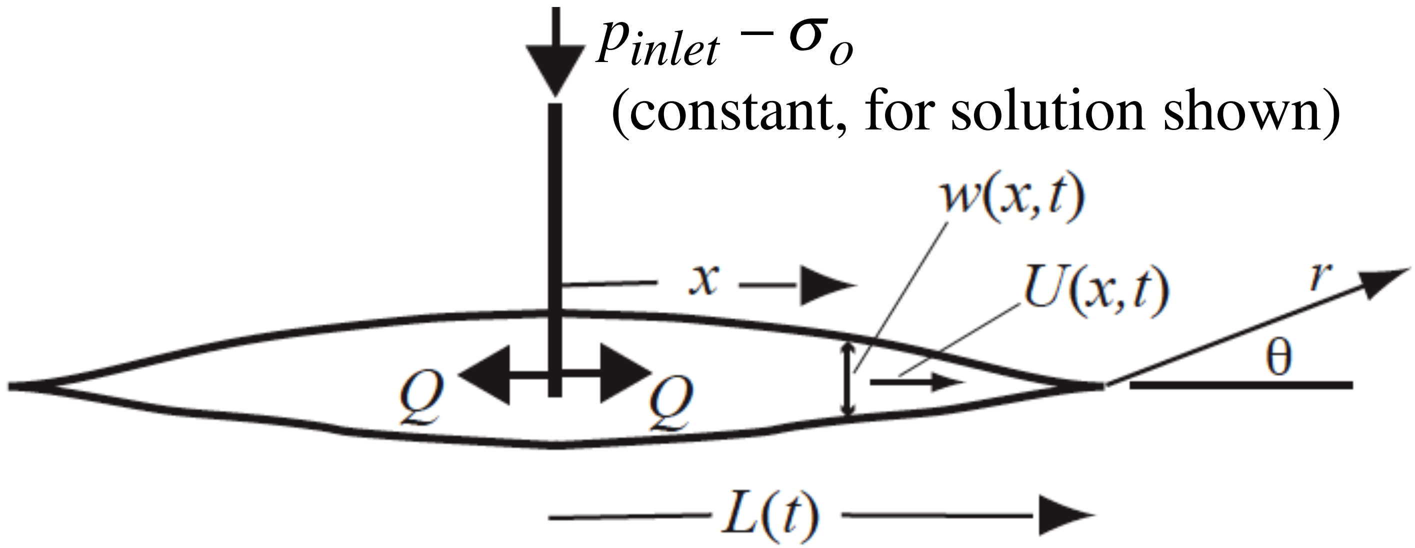



FIG. 1. Nikuradse's data. Up to a Re of about 3000 the flow is streamlined (free from turbulence) and $f \sim 1/\mathrm{Re}$. Note that for very rough pipes (small $R/r$) the curves do not form a belly at intermediate values of Re. Inset: verification of Strickler's empirical scaling for $f$ at high Re, $f \sim (r/R)^{1/3}$.

## *Exact Self-Similar Solution*

Approach similar to that of Adachi and Detournay (*Int. J. Numer. Anal. Meth. Geomech.*, 2002) for a viscous fluid in locally laminar flow:




***Their case*** (power-law viscous fluid): $\tau_{wall} = Const. \times \left(\dfrac{U}{\xi w}\right)^n = -\dfrac{1}{2}\xi w \dfrac{\partial p}{\partial x}$

***Our case*** (turbulent, high **Re**): $\tau_{wall} = \dfrac{f}{8}\rho U^2 = -\dfrac{1}{2}\xi w \dfrac{\partial p}{\partial x}$ , $f = 0.143\left(\dfrac{k}{\xi w}\right)^{1/3}$

$$L(t) = C\, t^{6/5}, \quad w(x,t) \sim C\, t^{6/5} F(x/L(t)),$$

$$p(x,t) - p_o = G\,(x/L(t)), \quad U(x,t) = C\, t^{1/5} H(x/L(t)).$$

Elastically related (term by term) crack opening and pressure fields, with correct singularity at the moving fracture tip ($c_1$, $c_2$, *etc*. are chosen to make $K_I = 0$ for arbitrary values of unknown constants $D$, $A_1$, $A_2$, *etc*.):

$$\hat{w} = D\left[\frac{1}{\delta}\left(\frac{1-\hat{x}^2}{2}\right)^{6/7} + A_1 w_1(\hat{x}) + A_2 w_2(\hat{x}) + A_3 w_3(\hat{x}) + \dots\right]$$

$$\hat{p} = D\left[F(\hat{x}) + A_1(c_1 - |\hat{x}|) + A_2(c_2 - \hat{x}^2) + A_3(c_3 - |\hat{x}|^3) + \dots\right]$$

To obtain $D$, $A_1$, $A_2$, *etc*., and yet another constant $\phi$ which enters, we must demand that the fluid flow equation

$$-\hat{w}^{10/3}\frac{d\hat{p}}{d\hat{x}} = \frac{(6/5)^{1/3} f_0}{4\xi^{4/3}}\phi^2\left(\hat{x}\hat{w} + 2\int_{\hat{x}}^{1}\hat{w}(\hat{s})d\hat{s}\right)^2$$

be satisfied. That is done numerically, in a least squares sense.

(*Note:* $\hat{U}\hat{w} = \hat{x}\hat{w} + 2\int_{\hat{x}}^{1}\hat{w}(\hat{s})d\hat{s}$ )

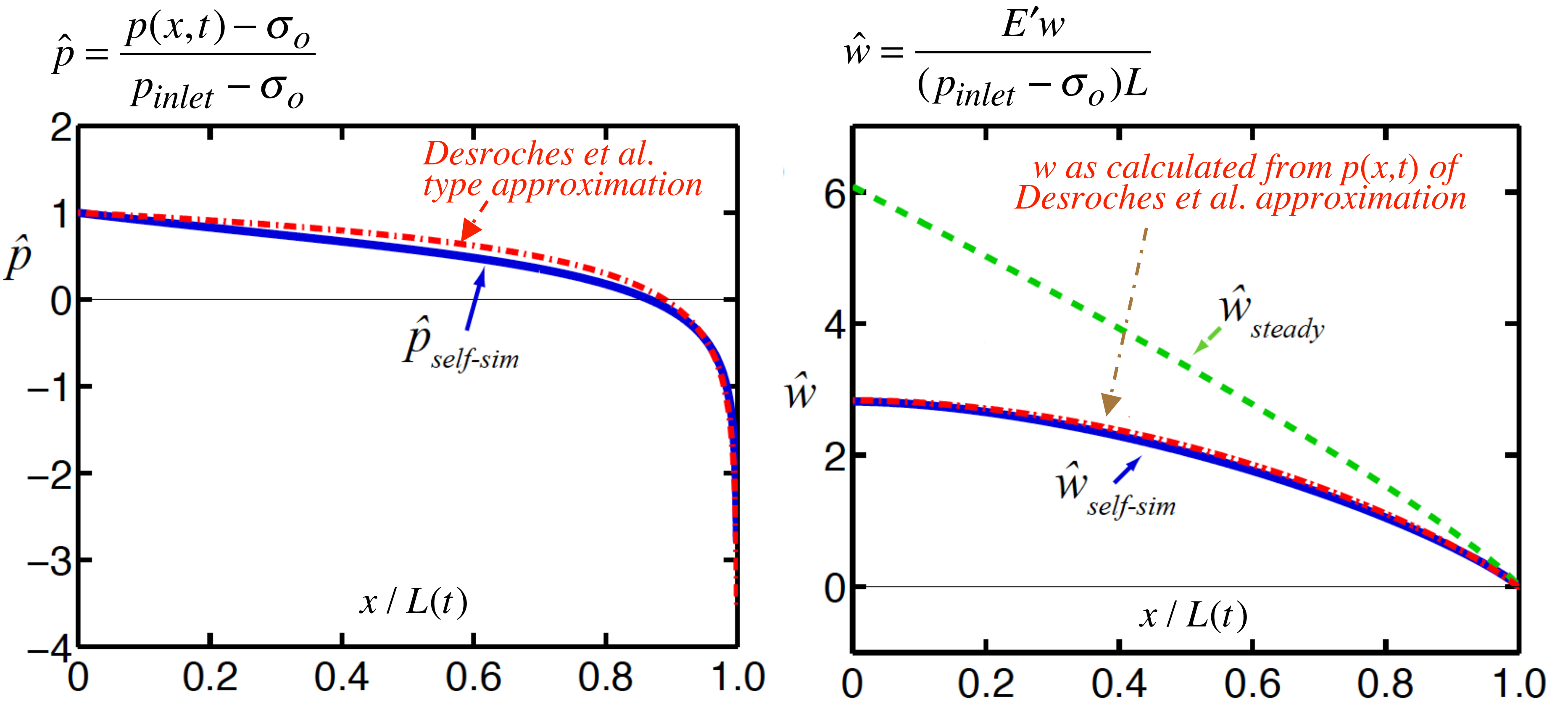
Solution (numerical, using superposition of exact elastic solutions with $K_I = 0$, and least-squares satisfaction of fluid flow equation):

$$L(t) = 5.74\left(\frac{p_{inlet} - \sigma_o}{E'}\right)^{4/5}\left[\left(\frac{p_{inlet} - \sigma_o}{\rho}\right)^{1/2} t\right]^{6/5} \frac{1}{k^{1/5}} \ .$$

$$\text{Also,}\quad h_{avg}(t) = 1.02\,\frac{p_{inlet} - \sigma_o}{E'}\,L(t) \ .$$

Using $\dfrac{dL/dt}{L} = \dfrac{6}{5t}$ to eliminate $t$ ,

$$\frac{dL}{dt} \equiv U_{tip} = 5.14\left(\frac{p_{inlet} - \sigma_o}{\rho}\right)^{1/2}\left(\frac{p_{inlet} - \sigma_o}{E'}\right)^{2/3}\left(\frac{L}{k}\right)^{1/6} .$$



$\hat{p} = \frac{p(x,t) - \sigma_o}{p_{inlet} - \sigma_o}$
$\hat{p}$
Desroches et al. type approximation
$\hat{p}_{self\text{-}sim}$
$x / L(t)$
2
1
0
−1
−2
−3
−4
0
0.2
0.4
0.6
0.8
1.0
$\hat{w} = \frac{E'w}{(p_{inlet} - \sigma_o)L}$
$\hat{w}$
w as calculated from p(x,t) of Desroches et al. approximation
$\hat{w}_{steady}$
$\hat{w}_{self\text{-}sim}$
$x / L(t)$
6
4
2
0
0
0.2
0.4
0.6
0.8
1.0

***Making contact with the observations [Das et al., '08] of surface-lake drainage driving hydraulic fracture near a margin of the Greenland Ice Sheet, and using analytical results for self-similar plane strain fracture:***

$E' = 6.8$ GPa, $\rho = 10^3$ kg/m$^3$, $p_{hydrostatic} - \sigma_o = 0.87$ MPa , $k = 1$ cm, $L = 1$ km:

- **If $p_{inlet} - \sigma_o = p_{hydrostatic} - \sigma_o$ , $U_{tip} = 2.6$ m/s , $h_{avg} = 13$ cm**

- **Half of $p_{hydrostatic} - \sigma_o$ lost on way to base: $U_{tip} = 1.2$ m/s , $h_{avg} = 7$ cm.**

- ***k* decreased by factor of 5, to $k = 2$ mm: $U_{tip} = 3.4$ m/s , $h_{avg} = 13$ cm.**

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- Young's modulus $E = 6.2$ GPa at –5ºC [Jellinek et al., '55] and Poisson's ratio $\nu = 0.3$ [Vaughan, '95], gives $E' = 6.8$ GPa.
- Liquid density $\rho = 1000$ kg/m$^3$, ice density $\rho_{ice} = 910$ kg/m$^3$.
- Ice thickness $H = 980$ m [Das et al., '08], so if *no head loss* along water column to glacial surface, $p_{inlet} - \sigma_o = 0.87$ MPa.
- Dependence of $U$ on channel wall roughness $k$ is weak (power law exponent = 1/6); estimate $k = 1$ cm, which is consistent with $n_{\text{Manning}} \sim 0.018$ s m$^{-1/3}$.

## ***Preliminary Greenland lake drainage rate comparisons:***

Assuming

- $p = p_{hydrostatic}$ at base
- $k = 1$ cm as earlier,
- width in orthogonal direction is $W = 3$ km (~ length of observed surface-breaking crevasse/moulin system),

predicted meltwater inflow rate $Q$ to the glacier bed is

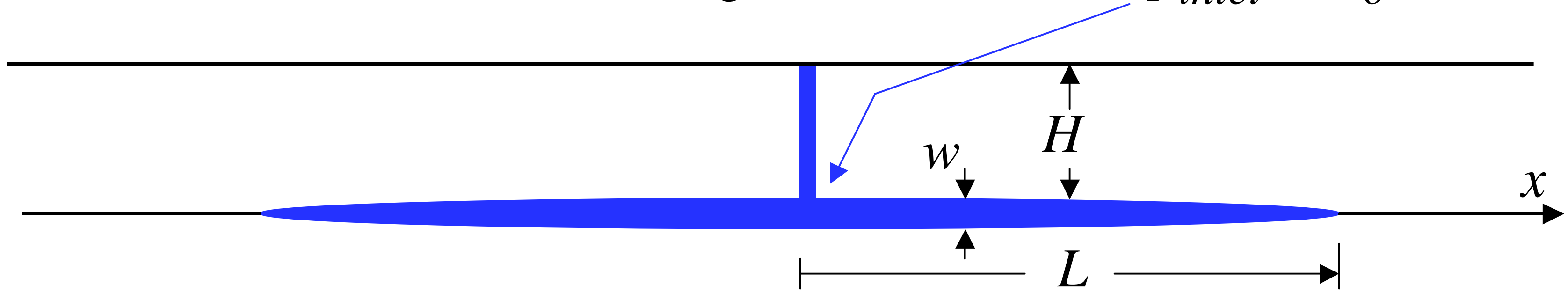
$$Q = W\frac{d}{dt}(2Lh_{avg}) = 4Wh_{avg}\frac{dL}{dt} \approx 21\,WL\left(\frac{p_{inlet} - \sigma_o}{\rho}\right)^{1/2}\left(\frac{p_{inlet} - \sigma_o}{E'}\right)^{5/3}\left(\frac{L}{k}\right)^{1/6}$$

giving $Q = 4.1\text{x}10^3$ m$^3$/s when $L = 1$ km;
$Q = 9.2\text{x}10^3$ m$^3$/s when $L = 2$ km.

Same order of magnitude as measured average rapid lake drainage rate
$Q = 8.7\text{x}10^3$ m$^3$/s ,
but results are preliminary: • ***Better analysis* needed for the range $L \sim H$ and $L > H$.** • ***Hydraulic head loss in flow to the bed must be evaluated.*** • ***Asymptotic use of plate theory for L >> H.*** • ***Viscoelastic material response.*** • ***Till entrainment.***

**Large $L$ / $H$ range**

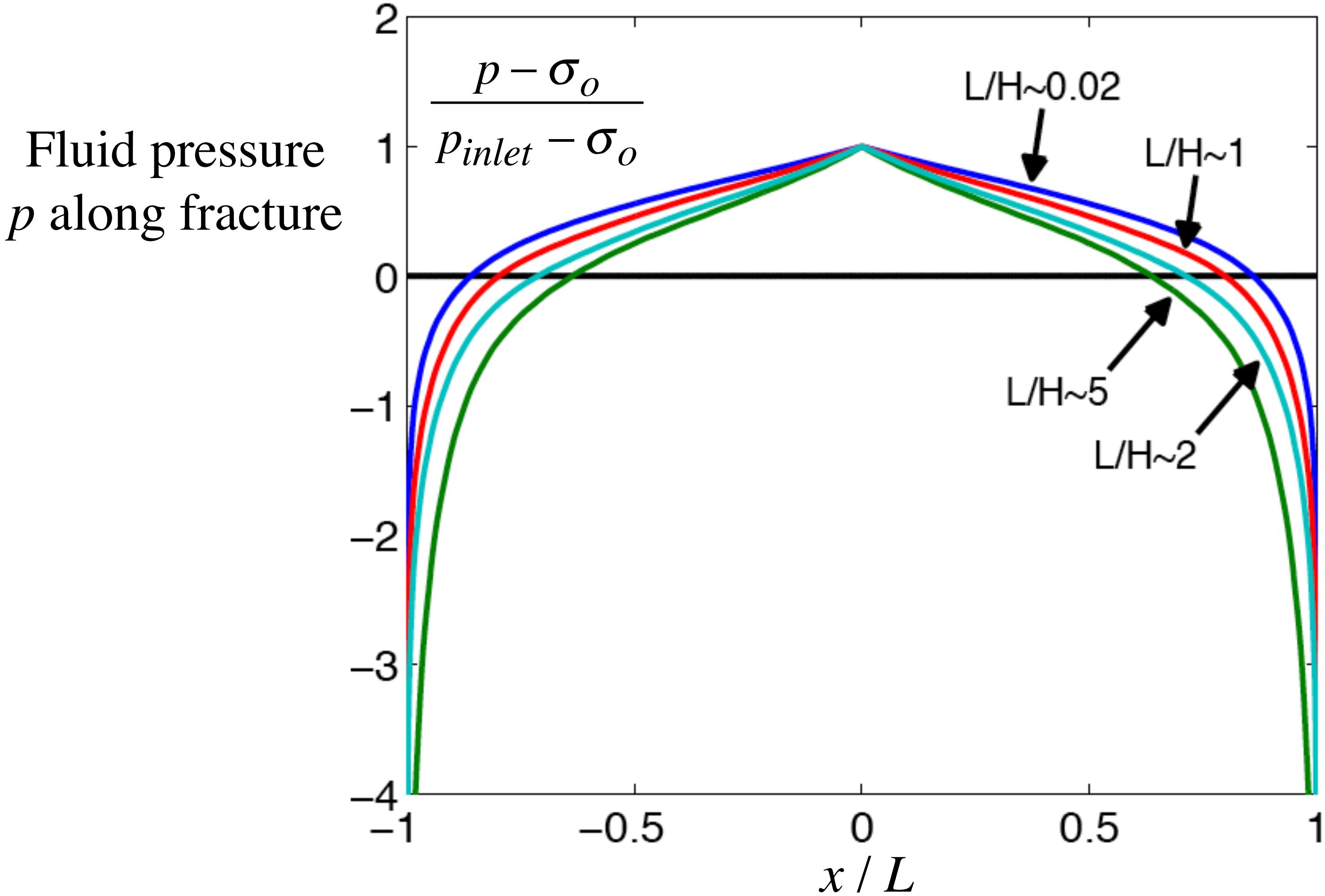
Results shown for fracture growth with fixed $p_{inlet} - \sigma_o$



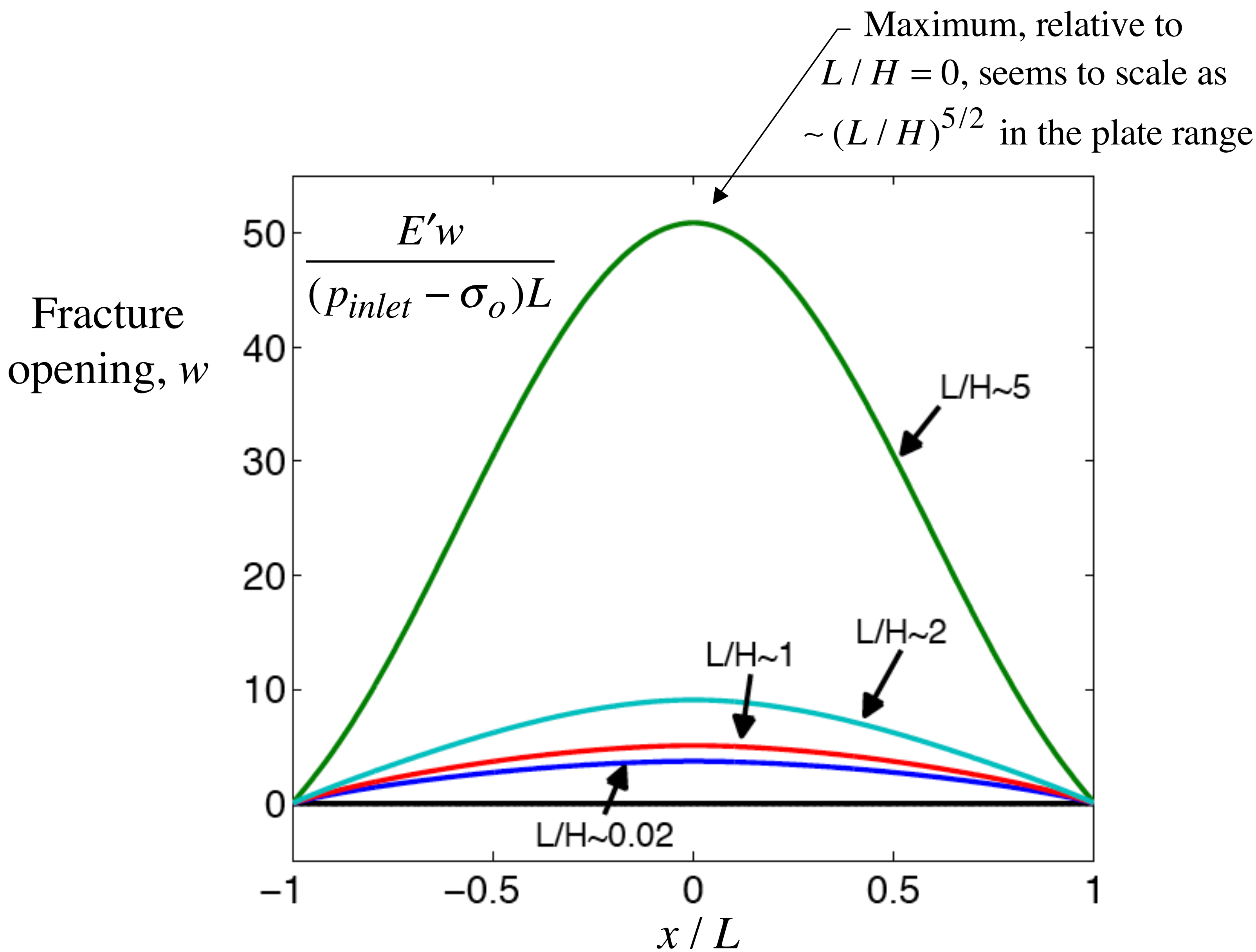

Elasticity part of the analysis based on Erdogan et al. (1973) integral relation, with corrected stress kernel ***k*(*x*,*s*;*H*)** (Head, 1953; Dmowska & Kostrov, 1973; Freund & Barnett, 1976) for opening dislocation near traction-free boundary of a half space:

$$p(x,t) - \sigma_o = \frac{E'}{4\pi} \int_{-L(t)}^{+L(t)} \left( \frac{1}{x-s} + k(x,s;H) \right) \frac{\partial w(s,t)}{\partial s} ds$$

(Analogous to zero $K_{\mathrm{Ic}}$ results of Zhang et al. (2005), but here applied to turbulent rather than laminar flow.)

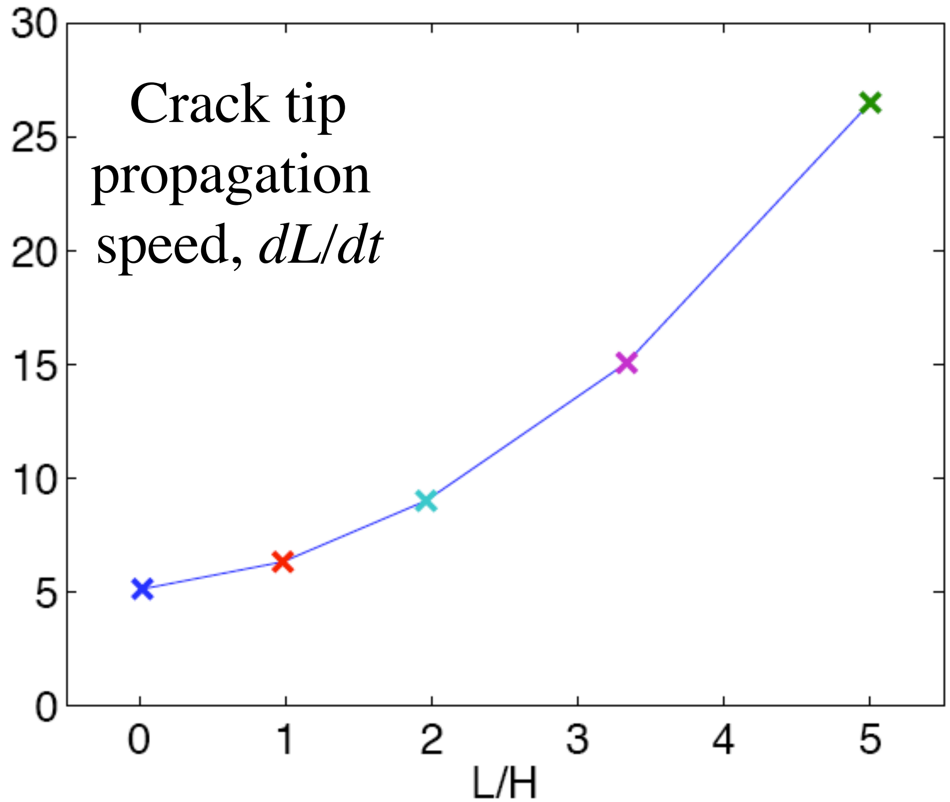


Fluid pressure
$p$ along fracture
$\frac{p-\sigma_o}{p_{inlet}-\sigma_o}$
L/H~0.02
L/H~1
L/H~5
L/H~2
2
1
0
−1
−2
−3
−4
−1
−0.5
0
0.5
1
$x\,/\,L$




Actual increase in $w$ likely to be much smaller, increased fracture volume means increased $Q$, $\Rightarrow$ more head loss to base, smaller $p_{inlet} - \sigma_o$

$$\frac{dL\,/\,dt}{\left(\frac{p_{inlet}-\sigma_o}{\rho}\right)^{1/2}\left(\frac{p_{inlet}-\sigma_o}{E'}\right)^{2/3}\left(\frac{L}{k}\right)^{1/6}}$$




Actual increase in $dL\,/\,dt$ likely to be much smaller, increased inflow rate means increased $Q$, $\Rightarrow$ more head loss to base, smaller $p_{inlet}-\sigma_o$

***Abstract:*** Meltwater generated at the surface and base of glaciers and ice sheets is known to have a large impact on how ice masses behave dynamically, but much is still unknown about the physical processes responsible for how this meltwater drains out of the glacier. For example, little attention has been paid to short-timescale processes like turbulent hydraulic fracture, which is likely an important mechanism by which drainage channels initially form when water pressures are high. In recent work (Tsai and Rice [Fall AGU, 2008; JGR subm., 2009]), we have constructed a model of this turbulent hydraulic fracture process in which over-pressurized water is assumed to flow turbulently through a crack, leading to crack growth. However, one important limitation of this prior work is that it only strictly applies in the limit of short crack length 2L compared to glacier height H, whereas relevant observations of supraglacial lake drainage, jokulhlaups and sub-glacial lake-to-lake transport episodes do not fall in this regime. Here, we improve somewhat upon this model by explicitly accounting for a nearby free surface. We accomplish this by applying the approach of Erdogan et al. [Meth. Anal. Sol. Crack Prob., 1973] to numerically calculate elastic displacements consistent with crack pressure distribution for a crack near a free surface, and use these results as before to simultaneously satisfy the governing fluid, elastic and fracture equations. Our results are analogous to the zero fracture toughness results of Zhang et al. [Int. J. Numer. Anal. Meth. Geomech., 2005], but applied to the case of turbulent flow rather than laminar flow of a Newtonian viscous fluid. Our new results clarify the importance of the free surface and potentially explain discrepancies between our previous modeling results and observations of supraglacial lake drainage by Das et al. [Science, 2008]. However, the numerical challenges increase as 2L becomes comparable to or much larger than H. We hope to ultimately develop simpler analyses for that range which make use of (visco)elastic plate theory at positions along the uplifted ice sheet that are remote from the fracturing front. This approach may also be of interest for tidal interactions with the ice-shelf grounding line location.